Differential Equations

A differential equation is an equation involving an unknown function and its derivative(s). Most any physical situation can be represented by a (system of) differential equation(s), maybe using more than one variable.

Initial / Final Conditions

If your function f is defined on an interval $[a,b]$, an initial condition is an equation of the form $\boxed{f(a)=y_0}$ for some constant y_0.

A final condition is of the form $\boxed{f(b)=y_1}$ for some constant y_1.

Example 1: (Forensics)

$$\frac{d\theta}{dt} = -k(\theta(t) - T)$$

constants

$\theta(t)$ = temperature of an object

T = ambient temperature (assumed constant)

k = constant of proportionality (experimetally determined)

between the rate of change of the surface temperature of the object and the difference of the temp. with the environment.

How to solve:

Let $T = 72^\circ F$

$K = \frac{1}{2}$

Our equation becomes

$$\frac{d\theta}{dt} = -\frac{1}{2}(\theta(t) - 72).$$

Divide both sides by

$$(\theta(t) - 72).$$

We get

$$\frac{1}{\theta(t)-72}\frac{d\theta}{dt} = -\frac{1}{2}.$$

Integrate both sides with respect to t.

$$\int \frac{1}{\theta(t)-72}\frac{d\theta}{dt}\,dt = \int -\frac{1}{2}\,dt$$

$$= -\frac{t}{2} + C.$$

Let $u = \theta(t)-72$, $du = \frac{d\theta}{dt}\,dt$.

The equation becomes

$$\int \frac{1}{u}\,du = -\frac{t}{2} + C.$$

Integrate!

$$\ln(|u|) = -\frac{t}{2} + C.$$

$u = \theta(t) - 72$, so

$$\ln(|\theta(t) - 72|) = -\frac{t}{2} + C.$$

Exponentiate both sides with e.

$$\cancel{e}^{\cancel{\ln}(|\theta(t)-72|)} = e^{-\frac{t}{2}+C}$$

$$|\theta(t)-72| = e^{-\frac{t}{2}+C}$$

If we know $\theta(t) \geq 72$, then we drop absolute values to get

$$\theta(t)-72 = e^{-\frac{t}{2}+C}$$

$$\boxed{\theta(t) = e^{-\frac{t}{2}+C} + 72}$$

Suppose $\theta(0) = 98.6^\circ$ F.

Then $(t=0)$

$$98.6 = e^c + 72$$

$$e^c = 26.6$$

$$\boxed{c = \ln(26.6)}$$

L'Hopital's Rule

(section 6.8)

What is the most annoying limit?

$$\frac{0}{0}$$

L'Hopital's rule takes care of this case.

L' Hopital's Rule $(a \neq \pm\infty)$

Let f, g be differentiable in an open interval containing $x=a$.
Suppose $g'(x) \neq 0$ on that interval (except maybe at $x=a$).

Then if either $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

- or - $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to\infty} g(x) = \pm\infty$,

then $\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$

Why should this be true?

Fake Reasoning:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f(x)}{g(x)}\cdot\underbrace{\frac{x-a}{x-a}}_{=1}$$

$$= \lim_{x\to a}\frac{f(x)}{x-a}\cdot\frac{x-a}{g(x)}$$

$$= \lim_{x\to a}\frac{f(x)}{x-a}\cdot\lim_{x\to a}\frac{1}{\left(\frac{g(x)}{x-a}\right)}$$

$$= f'(a)\cdot\frac{1}{g'(a)}$$

$$= \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Also works for limits to $\pm\infty$.

When to use it?

Indeterminate Forms

$$\boxed{\frac{0}{0},\ \frac{\pm\infty}{\pm\infty},\ \pm\infty\cdot 0,\ \infty-\infty,\ 1^{\pm\infty},\ 0^0,\ \infty^0}$$

Example 2: $\left(\frac{0}{0}\right)$

$$\lim_{x\to 1} \frac{13x^2+26x-39}{-5(x+1)^2+20}$$

Use l'Hopital's rule, denote use by " $\overset{l'H}{=}$ " (always note when you use this rule!)

$$\lim_{x\to 1} \frac{13x^2+26x-39}{-5(x+1)^2+20} \overset{l'H}{=} \lim_{x\to 1} \frac{26x+26}{-10(x+1)}$$

$$= \frac{52}{-20} = \boxed{-\frac{13}{5}}$$

Example 3: $(0 \cdot \pm\infty)$

$$\lim_{x\to\infty}\left(x^2 e^{-\sqrt{x}}\right).$$

Choose one factor to move to the denominator.

$$e^{-\sqrt{x}} = \frac{1}{\left(\frac{1}{e^{-\sqrt{x}}}\right)} = \frac{1}{e^{\sqrt{x}}}.$$

$$\lim_{x\to\infty}\left(x^2 e^{-\sqrt{x}}\right) = \lim_{x\to\infty}\frac{x^2}{e^{\sqrt{x}}}$$

$$\lim_{x\to\infty} \frac{x^2}{e^{\sqrt{x}}} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{2x}{e^{\sqrt{x}} \cdot \frac{1}{2}\left(x^{-1/2}\right)}$$

$$= \lim_{x\to\infty} \frac{2x \cdot 2 \cdot x^{1/2}}{e^{\sqrt{x}}}$$

$$= \lim_{x\to\infty} \frac{4x^{3/2}}{e^{\sqrt{x}}} = \frac{\infty}{\infty}$$

$$\overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{6x^{1/2}}{e^{\sqrt{x}} \cdot \frac{1}{2}\left(x^{-1/2}\right)}$$

$$= \lim_{x\to\infty} \frac{12x}{e^{\sqrt{x}}}$$

$$\lim_{x\to\infty} \frac{12x}{e^{\sqrt{x}}} = \frac{\infty}{\infty}$$

$$\overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{12}{e^{\sqrt{x}} \cdot \frac{1}{2}\left(x^{-1/2}\right)}$$

$$= \lim_{x\to\infty} \frac{24\, x^{1/2}}{e^{\sqrt{x}}} = \frac{\infty}{\infty}$$

$$\overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{24 \cdot \cancel{1/2}\,\cancel{\left(x^{-1/2}\right)}}{e^{\sqrt{x}} \cdot \cancel{1/2}\,\cancel{\left(x^{-1/2}\right)}}$$

$$= \lim_{x\to\infty} \frac{24}{e^{\sqrt{x}}} = \boxed{0}$$